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ASYMPTOTICS FOR AMITSUR'S CAPELLI-TYPE POLYNOMIALS AND VERBALLY PRIME PI-ALGEBRAS

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ABSTRACT

We consider associative PI-algebras over a field of characteristic zero. The main goal of the paper is to prove that the codimensions of a verbally prime algebra [11] are asymptotically equal to the codimensions of the T-ideal generated by some Amitsur's Capelli-type polynomials $E^*_{M,L}$ [1]. We recall that two sequences a_n, b_n are asymptotically equal, and we write $a_n \simeq b_n$, if and only if $\lim_{n\to\infty}(a_n/b_n) = 1$. In this paper we prove that

$$c_n(M_k(G)) \simeq c_n(E^*_{k^2,k^2}) \quad \text{and} \quad c_n(M_{k,l}(G)) \simeq c_n(E^*_{k^2+l^2,2kl}),$$

where G is the Grassmann algebra. These results extend to all verbally prime PI-algebras a theorem of A. Giambruno and M. Zaicev [9] giving the asymptotic equality

$$c_n(M_k(F)) \simeq c_n(E^*_{k^2,0})$$

between the codimensions of the matrix algebra $M_k(F)$ and the Capelli polynomials.

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1. Introduction

Let F be a field of characteristic zero and let $F\langle X\rangle$ be the free associative algebra over F of countable rank on the set $X = \{x_1, x_2, \ldots\}$. Recall that an ideal I of $F\langle X\rangle$ is a T-ideal if it is invariant under all endomorphisms of $F\langle X\rangle$.

Let A be an associative algebra over F; an element $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$ is called a polynomial identity for A if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. If f is a polynomial identity for A we usually write $f \equiv 0$ in A. Let $T(A) = \{f \in F\langle X\rangle\colon f \equiv 0 \text{ in } A\}$ be the ideal of polynomial identities of A. When A satisfies a non-trivial identity (i.e. $T(A) \neq (0)$), we say that A is a PI-algebra.

The connection between T-ideals of $F\langle X\rangle$ and PI-algebras is well understood: For any F-algebra A, $T(A)$ is a T-ideal of $F\langle X\rangle$ and every T-ideal I of $F\langle X\rangle$ is the ideal of identities of some F-algebra A, $I = T(A)$.

For $I = T(A)$ a T-ideal of $F\langle X\rangle$, we denote by $\mathrm{var}(I)$ or $\mathrm{var}(A)$ the variety of all associative algebras having the elements of I as polynomial identities.

An important class of T-ideals is given by the so-called verbally prime T-ideals. They were introduced by Kemer (see [11]) in his solution of the Specht problem as basic blocks for the study of arbitrary T-ideals. Recall that a T-ideal $I \subsetneq F\langle X\rangle$ is verbally prime if $f(x_1, \ldots, x_r)g(x_{r+1}, \ldots, x_n) \in I$ implies that either $f \in I$ or $g \in I$. A PI-algebra A is called verbally prime if its T-ideal of identities $I = T(A)$ is verbally prime. Also, the corresponding variety of associative algebras $\mathrm{var}(A)$ is called verbally prime. By the structure theory of T-ideals developed by Kemer (see [11]) and his classification of verbally prime T-ideals in characteristic zero, the study of an arbitrary T-ideal in characteristic zero can be reduced to the study of the T-ideals of identities of the following verbally prime algebras:

$$F, F\langle X\rangle, M_k(F), M_k(G), M_{k,l}(G) \quad (k > 0, l > 0),$$

where $G = G^{(0)} \dot{+} G^{(1)}$ is the infinite-dimensional Grassmann algebra, $M_k(F)$, $M_k(G)$ are the algebras of $k \times k$ matrices over F and G, respectively, and

$$M_{k,l}(G) = \begin{array}{c} \\ k \\ l \end{array}\!\!\begin{array}{c} \begin{array}{cc} k & \;\; l \end{array} \\ \begin{pmatrix} G^{(0)} & G^{(1)} \\ G^{(1)} & G^{(0)} \end{pmatrix} \end{array}.$$

Recall that G is the algebra generated by a countable set $\{e_1, e_2, \ldots\}$ subject to the conditions $e_ie_j = -e_je_i$ for all $i, j = 1, 2, \ldots$, and $G = G^{(0)} \dot{+} G^{(1)}$ is the natural $\mathbb{Z}_2$-grading on G, where $G^{(0)}$ and $G^{(1)}$ are the spaces generated by all monomials in the generators e_i of even and odd length, respectively.

It is well known that in characteristic zero every T-ideal is completely determined by its multilinear elements. Hence, if P_n is the space of multilinear polynomials of degree n in $x_1, \dots, x_n$, we study the sequence of spaces $P_n \cap T(A)$, $n = 1, 2, \dots$.

A useful approach to this study is through the representation theory of the symmetric group S_n. In fact, there is a natural action of S_n on P_n leaving $P_n \cap T(A)$ invariant: if $\sigma \in S_n$ and $f(x_1, \dots, x_n) \in P_n$ then one defines $\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. This in turn makes $P_n(A) = P_n/(P_n \cap T(A))$ an S_n-module.

The S_n-character of $P_n(A)$, denoted by $\chi_n(A)$, is called the n-th cocharacter of A or of $T(A)$. By complete reducibility, $\chi_n(A)$ decomposes into irreducibles and let $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$, where χ_λ is the irreducible S_n-character associated to the partition λ of n and m_λ is the corresponding multiplicity. Through the sequence of cocharacters $\{\chi_n(A)\}_{n \geq 1}$ one can attach to A a numerical sequence called the sequence of codimensions $\{c_n(A)\}_{n \geq 1}$ of I or A, where

$$c_n(A) = \chi_n(A)(1) = \dim_F P_n/(P_n \cap T(A)),$$

$n = 1, 2, \dots$.

It is clear that A is a PI-algebra if and only if $c_n(A) < n!$ for some $n \geq 1$. Regev in [12] proved that if A is an associative PI-algebra, then $c_n(A)$ is exponentially bounded. Hence there exist constants α, β such that $c_n(A) \leq \alpha\beta^n$ for any $n \geq 1$. It was recently proved by Giambruno and Zaicev, in [6] and [7], that for a PI-algebra A

$$\exp(A) = \lim_{n \to \infty} \sqrt[n]{c_n(A)}$$

exists and is an integer; $\exp(A)$ is called the PI-exponent of the algebra A. For the verbally prime algebras we have

$$\exp(M_k(F)) = k^2, \quad \exp(M_k(G)) = 2k^2, \quad \exp(M_{k,l}(G)) = (k+l)^2.$$

These results were first proved in [14], [15]. Improved proofs appeared later in [2], [7].

In [14] Regev obtained the precise asymptotic behavior of the codimensions of the verbally prime algebra $M_k(F)$. It turns out that

$$c_n(M_k(F)) \simeq C\left(\frac{1}{n}\right)^{(k^2-1)/2} k^{2n},$$

where C is a certain constant explicitly computed. For the other verbally prime PI-algebras $M_k(G)$, $M_{k,l}(G)$ there are only some partial results (see [2]).

It turns out that it is in general a very hard problem to determine the precise asymptotic behavior of such sequences.

In this paper we find a relation among the asymptotics of the codimensions of the verbally prime T-ideals and the T-ideals generated by Amitsur's Capelli-type polynomials.

Now, if $f \in F\langle X\rangle$ we denote by $\langle f\rangle_T$ the T-ideal generated by f. Also for $V \subset F\langle X\rangle$ we write $\langle V\rangle_T$ to indicate the T-ideal generated by V.

Let L and M be two natural numbers, let $\hat{n} = (L+1)(M+1)$ and let μ be a partition of $\hat{n}$ with associated rectangular Young diagram, $\mu = ((L+1)^{M+1}) \vdash \hat{n}$. In [1] the following polynomials were introduced denoting Amitsur's Capelli-type polynomials:

$$\begin{aligned} e^*_{M,L}(\overline{x},\overline{y}) &= e^*_{M,L}(x_1,\dots,x_{\hat{n}};y_1,\dots,y_{\hat{n}-1}) \\ &= \sum_{\sigma\in S_{\hat{n}}} \chi_\mu(\sigma)x_{\sigma(1)}y_1x_{\sigma(2)}\cdots y_{\hat{n}-1}x_{\sigma(\hat{n})}, \end{aligned}$$

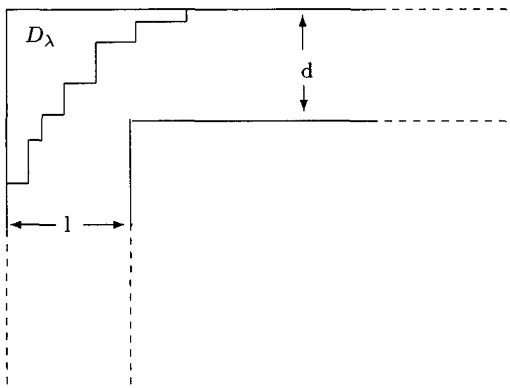
where $\chi_\mu(\sigma)$ is the value of the irreducible character χ_μ corresponding to the partition $\mu \vdash \hat{n}$ on the permutation σ. We note that for $L = 0$ we have $\mu = (1^{\hat{n}})$ and

$$e^*_{M,L}(\overline{x},\overline{y}) = c_{\hat{n}}(\overline{x},\overline{y}) = \sum_{\sigma\in S_{\hat{n}}} (\operatorname{sgn}\sigma)x_{\sigma(1)}y_1x_{\sigma(2)}\cdots y_{\hat{n}-1}x_{\sigma(\hat{n})}$$

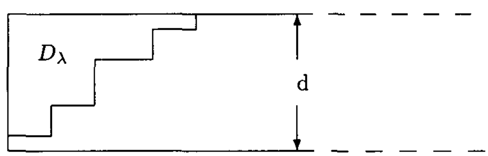
is the Capelli polynomial.

Amitsur's Capelli-type polynomials generalize the Capelli polynomials in the sense that the Capelli polynomials characterize the algebras having the cocharacter contained in a given strip (see [13]) and Amitsur's polynomials characterize the algebras having a cocharacter contained in a given hook (see [1, Theorem B]).

More precisely, given any integer $d, l \geq 0$ we denote by $H(d,l) = \bigcup_{n\geq 1}\{\lambda = (\lambda_1,\lambda_2,\dots) \vdash n\colon \lambda_{d+1} \leq l\}$ the infinite hook of arm d and leg l. If the partition λ lies in $H(d,l)$ then its corresponding Young diagram D_λ is contained in the (d,l) hook.



Note that $H(d,0)$ is the set of all partitions with diagrams contained in the strip of height d.



Regev proved (see [13, Theorem 2]) that, if A is a PI-algebra, then A satisfies the Capelli identity $c_d \equiv 0$ if and only if $\chi_n(A) = \sum_{\substack{\lambda\vdash n,\\ \lambda\in H(d-1,0)}} m_\lambda\chi_\lambda$. This result characterizes the cocharacter sequence of those PI-algebras satisfying a Capelli polynomial. Thus the Capelli identities can be used as a test for a PI-algebra to have cocharacter sequence lying in a strip.

Generalizing this approach Amitsur and Regev proved that the Capelli-type polynomials $e^*_{M,L}$ characterize the algebras whose cocharacter sequence lies in the hook $H(M,L)$. More precisely, if A is a PI-algebra, then A satisfies the Capelli-type identity $e^*_{M,L} \equiv 0$ if and only if $\chi_n(A) = \sum_{\substack{\lambda\vdash n,\\ \lambda\in H(M,L)}} m_\lambda\chi_\lambda$ (see [1, Theorem B]).

Let $E^*_{M,L}$ denote the set of $2^{\hat{n}-1}$ polynomials obtained from $e^*_{M,L}$ by evaluating the variables y_i to 1 in all possible ways. Also, we denote by $\Gamma_{M,L} = \langle E^*_{M,L}\rangle_T$ the T-ideal generated by $E^*_{M,L}$. We also write $\mathcal{V}_{M,L} = \mathrm{var}(E^*_{M,L}) = \mathrm{var}(\Gamma_{M,L})$, $c_n(E^*_{M,L}) = c_n(\Gamma_{M,L})$ and $\exp(E^*_{M,L}) = \exp(\Gamma_{M,L})$.

The following relations between the exponent of the Capelli-type polynomials and the exponent of the verbally prime algebras are well known (see [3], [7]):

$$\exp(E^*_{k^2,0}) = k^2 = \exp(M_k(F)),$$

$$\exp(E^*_{k^2,k^2}) = 2k^2 = \exp(M_k(G)),$$
$$\exp(E^*_{k^2+l^2,2kl}) = (k+l)^2 = \exp(M_{k,l}(G)).$$

Also in [9] it was proved that the codimensions of $\Gamma_{k^2,0}$ are asymptotically equal to the codimensions of the verbally prime algebra $M_k(F)$,

$$c_n(E^*_{k^2,0}) = c_n(C_{k^2+1}) \simeq c_n(M_k(F)).$$

In this paper we obtain an analogous result for the other verbally prime algebras. Namely, we prove the following asymptotic equalities:

$$c_n(E^*_{k^2,k^2}) \simeq c_n(M_k(G)) \quad \text{and} \quad c_n(E^*_{k^2+l^2,2kl}) \simeq c_n(M_{k,l}(G)).$$

2. Asymptotics for $E^*_{k^2+l^2,2kl}$ and $M_{k,l}(G)$

In this section we shall prove our main result about the Capelli-type polynomial $E^*_{k^2+l^2,2kl}$ where $k,l \in \mathbb{N}$, and the verbally prime algebra $M_{k,l}(G)$.

Throughout the paper we will denote by F a field of characteristic zero. Recall that an algebra A is a superalgebra (or $\mathbb{Z}_2$-graded algebra) with grading $(A^{(0)}, A^{(1)})$ if $A = A^{(0)} \dot{+} A^{(1)}$ is a direct sum as a space of its subspaces $A^{(0)}, A^{(1)}$ satisfying

$$A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)} \quad \text{and} \quad A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}.$$

If $G = G^{(0)} \dot{+} G^{(1)}$ is the infinite-dimensional Grassmann algebra over F, then $G(A) = A^{(0)} \otimes G^{(0)} \dot{+} A^{(1)} \otimes G^{(1)}$ is called the Grassmann envelope of A. We recall that, by a result of Kemer (see [11, Theorem 2.3]), if $\mathcal{V}$ is a proper variety then there exists a finite-dimensional superalgebra A such that $\mathcal{V} = \mathrm{var}(G(A))$. In what follows the symbol "$\oplus$" will denote a direct sum of algebras and the symbol "$\dot{+}$" will denote a direct sum of vector spaces.

The notion of reduced superalgebra was introduced in [9, Definition 1]. Let $A = A_1 \oplus \cdots \oplus A_r \dot{+} J$ be a finite-dimensional superalgebra with $A_1, \ldots, A_r$ simple superalgebras and $J = J(A)$ the Jacobson radical of A; A is called reduced if $A_1 J A_2 \cdots J A_r \neq 0$. Giambruno and Zaicev showed, also, that these superalgebras can be used as building blocks of any proper variety. They proved that (see [9, Theorem 1]) if $\mathcal{V}$ is a proper variety of algebras, then there exists a finite number of reduced superalgebras $B_1, \ldots, B_t$ and a finite-dimensional superalgebra D such that $\mathcal{V} = \mathrm{var}(G(B_1) \oplus \cdots \oplus G(B_t) \oplus G(D))$, where $\exp(\mathcal{V}) = \exp(G(B_1)) = \cdots = \exp(G(B_t))$ and $\exp(G(D)) < \exp(\mathcal{V})$.

Now we analyze the case of a reduced superalgebra of special type. Recall that $M_{k,l}(F)$ denotes the simple superalgebra of $(k+l)\times(k+l)$ matrices over F with grading $\left(\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}, \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix}\right)$, where F_{11}, F_{12}, F_{21}, F_{22} are $k\times k$, $k\times l$, $l\times k$ and $l\times l$ matrices, respectively.

Throughout this section we assume that $A = M_{k,l}(F)\dot{+}J$, where $J = J(A)$ is the Jacobson radical of the finite-dimensional superalgebra A. Note that $M_{k,l}(F)$ contains the unit and it certainly belongs to the even part in the grading. It is also known that J is homogeneous under the grading of A [11]. We start with the following key lemmas.

Lemma 1: *The Jacobson radical J can be decomposed into the direct sum of four $M_{k,l}(F)$-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $p,q\in\{0,1\}$, J_{pq} is a left faithful module or a 0-left module according as $p=1$ or $p=0$, respectively. Similarly, J_{pq} is a right faithful module or a 0-right module according as $q=1$ or $q=0$, respectively. Moreover, for $p,q,i,l\in\{0,1\}$, $J_{pq}J_{ql}\subseteq J_{pl}$, $J_{pq}J_{il}=0$ for $q\neq i$ and there exists a finite-dimensional nilpotent superalgebra N such that $J_{11}\cong M_{k,l}(F)\otimes_F N$ (isomorphism of $M_{k,l}(F)$-bimodules and of superalgebras).

Proof: The proof of the first part of the lemma is the same as that in [9, Lemma 2]. Now let $\{j_1,\dots,j_s\}$ be a basis of J_{11}. We can suppose that all elements j_q are homogeneous in the grading (either even or odd). Then $J_{11} = \mathrm{Span}\{e_{rs}je_{mt}\colon r,s,m,t = 1,\dots,k+l, j\in\{j_1,\dots,j_s\}\}$. If $d_{st}(j) = \sum_{i=1}^{k+l} e_{is}je_{ti}\in J_{11}$, then we put $N = \mathrm{Span}\{d_{st}(j)\colon s,t = 1,\dots,k+l, j\in\{j_1,\dots,j_s\}\}$. Notice that $e_{rs}je_{tm}$ has the same grading as $e_{st}je_{rm}$. Hence the grading of $d_{st}(j)$ is equal to the grading of j plus the grading of e_{st} modulo 2 and all $d_{st}(j)$ are homogeneous. Thus $N = N^{(0)}\oplus N^{(1)}$, where $N^{(0)}$ is generated by all elements $d_{st}(j)$ with grading zero and $N^{(1)}$ is generated by the elements with grading one. N commutes with $M_{k,l}(F)$; in fact, $e_{rm}d_{st}(j) = e_{rm}(\sum_{i=1}^{k+l} e_{is}je_{ti}) = e_{rm}e_{ms}je_{tm} = e_{rs}je_{tm}$ and $d_{st}(j)e_{rm} = (\sum_{i=1}^{k+l} e_{is}je_{ti})e_{rm} = e_{rs}je_{tr}e_{rm} = e_{rs}je_{tm}$. Moreover, if we define an F-linear map $\varphi\colon J_{11}\to M_{k,l}(F)\otimes N$ by $\varphi(e_{rs}je_{mt}) = e_{rt}\otimes d_{sm}(j)$, then it is easy to show that φ is an isomorphism of superalgebras. ∎

Lemma 2: *Let $M = k^2+l^2$ and $L = 2kl$ with $k,l\in\mathbb{N}$. If $E^*_{M,L}\subseteq Id(G(A))$, then $J_{10} = J_{01} = (0)$.*

Proof: First we shall determine a polynomial $e_1^*(\overline{x};\overline{y})$ which is a consequence of $E^*_{M,L}$ and then, by an opportune substitution of elements of $G(A)$ in $e_1^*(\overline{x};\overline{y})$, we shall obtain the conclusion of the lemma. Let $\lambda = ((L+1)^{M+1}) \vdash n$ be the partition of $n=(M+1)(L+1)$. Let us consider the following Young tableaux T_λ associated to the diagram D_λ,

$$T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 1+(M+1) & \cdots & 1+L(M+1) \\ \hline 2 & 2+(M+1) & \cdots & 2+L(M+1) \\ \hline \vdots & \vdots & & \vdots \\ \hline M+1 & 2(M+1) & \cdots & (L+1)(M+1) \\ \hline \end{array}$$

It is well known [10] that to T_λ one associates two subgroups of S_n:

$$R_{T_\lambda} = S_{L+1}(1, 1+(M+1), 1+2(M+1), \dots, 1+L(M+1)) \times \cdots \\ \times S_{L+1}(M+1, 2(M+1), 3(M+1), \dots, (L+1)(M+1))$$

and

$$C_{T_\lambda} = S_{M+1}(1,\dots,M+1) \times \cdots \times S_{M+1}(1+L(M+1),\dots,(L+1)(M+1)),$$

where $S_t(\beta_1,\dots,\beta_t)$ stands for the symmetric group of degree t on the elements $\beta_1,\dots,\beta_t$. R_{T_λ} (respectively C_{T_λ}) is the subgroup of S_n leaving the rows (respectively the columns) of T_λ invariant. The polynomial corresponding to T_λ will be

$$e_{T_\lambda}(\overline{x}) = \sum_{\rho \in R_{T_\lambda}} \rho g_{T_\lambda}(\overline{x})$$

where

$$g_{T_\lambda}(\overline{x}) = \prod_{i=1}^{L+1} \left(\sum_{\sigma_i \in S_{M+1}} (-1)^{\sigma_i} x_{\sigma_i(1+(i-1)(M+1))} \cdots x_{\sigma_i(M+1+(i-1)(M+1))} \right)$$

and $S_{M+1}(1+(i-1)(M+1),\dots,i(M+1))$ is the symmetric group of degree $M+1$ on the elements $1+(i-1)(M+1),\dots,i(M+1)$. Here, $g_{T_\lambda}(\overline{x})$ and $e_{T_\lambda}(\overline{x})$ are multilinear polynomials in $\overline{x} = \{x_1,\dots,x_n\}$. Moreover, $g_{T_\lambda}(\overline{x})$ is alternating on each set of variables $\widehat{x}_i = \{x_{1+(i-1)(M+1)},\dots,x_{M+1+(i-1)(M+1)}\}$ for $i=1,\dots,L+1$ and $e_{T_\lambda}(\overline{x})$ is symmetric on each set of variables $\widetilde{x}_i = \{x_i, x_{i+(M+1)},\dots,x_{i+L(M+1)}\}$ for $i=1,\dots,M+1$. Then, the polynomial

$$e_1^*(\overline{x};\overline{y}) = e^*_{T_\lambda}(\overline{x};\overline{y}) = \sum_{\rho \in R_{T_\lambda}} \rho g^*_{T_\lambda}(\overline{x};\overline{y})$$

where

$$g^*_{T_\lambda}(\overline{x};\overline{y}) = \prod_{i=1}^{L+1}\Bigg(\sum_{\sigma_i\in S_{M+1}}(-1)^{\sigma_i}\Bigg(\prod_{j=1}^{M+1} y_{j+(i-1)(M+1)}x_{\sigma_i(j+(i-1)(M+1))}\Bigg)\Bigg)$$

is multilinear in $\overline{x} = \{x_1,\dots,x_n\}$ and $\overline{y} = \{y_1,\dots,y_n\}$ and symmetric on each set of variables $\widetilde{x}_i$ for $i = 1,\dots,M+1$.

From [1] it follows that $e^*_1(\overline{x};\overline{y})$ is a consequence of $e^*_{M,L}(\overline{x};\overline{y})$. Since $E^*_{M,L}(\overline{x};\overline{y}) \subseteq Id(G(A))$, we have $e^*_1(\overline{x};\overline{y}) \in Id(G(A))$. Hence $e^*_1(\overline{sx};\overline{sy}) = 0$ for all substitutions of elements of $G(A)$, $\overline{sx} = \{\overline{sx}_1,\dots,\overline{sx}_n\}$ and $\overline{sy} = \{\overline{sy}_1,\dots,\overline{sy}_n\}$ with $\overline{sx}_i, \overline{sy}_j \in G(A)$ for $i = 1,\dots,n$ and $j = 1,\dots,n$.

Let now $e^0_1,\dots,e^0_M$ be an ordered basis of $M_{k,l}(F)^{(0)}$ consisting of all matrix units,

$$e^0_h \in \{e_{i,j} | 1\le i\le k, 1\le j\le k\} \cup \{e_{i,j} | k+1\le i\le k+l, k+1\le j\le k+l\}$$

and let $e^1_1,\dots,e^1_L$ be an ordered basis of $M_{k,l}(F)^{(1)}$ consisting of all matrix units from the odd part of $M_{k,l}(F)$,

$$e^1_h \in \{e_{i,j} | 1\le i\le k, k+1\le j\le k+l\} \cup \{e_{i,j} | k+1\le i\le k+l, 1\le j\le k\}.$$

Then we consider the following substitution:

$$\overline{sx}_{i+(j-1)(M+1)} := e^0_i \otimes g^0_{ji}, \quad j = 1,\dots,L+1,$$

for all $i = 1,\dots M$, where g^0_{ji} are all distinct elements from $G^{(0)}$,

$$\overline{sx}_{j(M+1)} := e^1_j \otimes g^1_{jM+1}, \quad j = 1,\dots,L,$$

with g^1_{jM+1} all distinct elements of $G^{(1)}$, and

$$\overline{sx}_{(L+1)(M+1)} := r_{10} \otimes g,$$

for any $g \in G$ and arbitrary $r_{10} \in J_{10}$. We take also

$$\overline{sy}_i := e_{hk} \otimes g_i,$$

for all $i = 1,\dots,n$, where $g_i \in G^{(0)} \cup G^{(1)}$ and e_{hk} are some opportune matrix units to fix the places.

By the properties of the polynomial $e^*_1(\overline{x};\overline{y})$ and the particular substitutions considered (recall that $G^{(0)} = Z(G)$), we have

$$e^*_1(\overline{sx};\overline{sy}) = ((L+1)!)^M (e_{ij}r_{10}) \otimes \overline{g} = 0,$$

where $\overline{g} \in G$, $\overline{g} \neq 0$. Then $e_{ij}r_{10} = 0$ for all $i, j \in \{1, \ldots, k+l\}$. Hence we can say that $r_{10} = 0$ for all $r_{10} \in J_{10}$ and the conclusion is obtained.

A similar proof shows that $J_{01} = (0)$. ∎

LEMMA 3: *Let $M = k^2 + l^2$ and $L = 2kl$ with $k, l \in \mathbb{N}$. Let $J_{11} \cong M_{k,l}(F) \otimes N$, where $N = N^{(0)} \dot{+} N^{(1)}$, as in Lemma 1. If $E^*_{M,L} \subseteq Id(G(A))$, then $N^{(0)} \subseteq Z(N)$, the center of N, and $N^{(1)}$ is anticommutative (or, which is the same, is nil of degree 2).*

Proof: We will construct a polynomial $e_2^*(\overline{x}; \overline{y})$ as in Lemma 2. Let $\mu = ((L+1)^{M+2})$ be a partition of $n' = (L+1)(M+2) = n + (L+1)$ and D_μ the corresponding Young diagram. As in Lemma 2 we consider the Young tableaux

$T_\mu =$

1	1+(M+2)	⋯	1+L(M+2)
2	2+(M+2)	⋯	2+L(M+2)
⋮	⋮		⋮
M+1	(M+1)+(M+2)	⋯	(M+1)+L(M+2)
M+2	2(M+2)	⋯	(L+1)(M+2)

and we determine the polynomial $e_2^*(\overline{x}; \overline{y}) = e^*_{T_\mu}(\overline{x}; \overline{y})$. Then we make a similar substitution as in Lemma 2: let $e_1^0, \ldots, e_M^0$ be an ordered basis of matrix units of $M_{k,l}(F)^{(0)}$ and $e_1^1, \ldots, e_L^1$ an ordered basis of matrix units of $M_{k,l}(F)^{(1)}$. We put

$$\overline{sx}_{i+(j-1)(M+2)} := e_i^0 \otimes g_{ji}^0, \quad j = 1, \ldots, L+1,$$

for all $i = 1, \ldots, M$, where $g_{ji}^0 \in G^{(0)}$ and they all depend on distinct generators,

$$\overline{sx}_{(M+1)+(j-1)(M+2)} := e_j^1 \otimes g_{jM+1}^1, \quad j = 1, \ldots, L$$

and

$$\overline{sx}_{j(M+2)} := e_j^1 \otimes g_{jM+2}^1, \quad j = 1, \ldots, L$$

where $g_{jM+1}^1, g_{jM+2}^1 \in G^{(1)}$ are all distinct,

$$\overline{sx}_{(M+1)+L(M+2)} := d_1 \otimes g_1 \quad \text{and} \quad \overline{sx}_{(L+1)(M+2)} := d_2 \otimes g_2,$$

where $g_1, g_2 \in G^{(0)} \cup G^{(1)}$ and $d_1, d_2 \in N^{(0)} \cup N^{(1)}$. Also we put

$$\overline{sy}_i := e_{hk} \otimes g_i, \quad i = 1, \ldots, n,$$

where $g_i \in G^{(0)} \cup G^{(1)}$ and e_{hk} are some opportune matrix units.

Note that $e_2^*(\overline{x}; \overline{y})$ is a multilinear polynomial in the set of variables $\overline{x}$ and $\overline{y}$; it has similar properties of symmetrizing and alternating as $e_1^*(\overline{x}; \overline{y})$ in Lemma 2. Now, we consider four different cases:

CASE 1: Let $d_1, d_2 \in N^{(0)}$. In this case by the same reasons as in [9, Lemma 4] for the Capelli polynomial, we obtain

$$e_2^*(\overline{sx}, \overline{sy}) = ((L+1)!)^M \cdot 2^L \cdot \binom{k+l+2}{2} \cdot [(e_{ij}[d_1, d_2]) \otimes g] = 0,$$

for some $g \in G$, for any $i, j \in \{1, \ldots, k+l\}$. Then $[d_1, d_2] = 0$, for all $d_1, d_2 \in N^{(0)}$.

CASE 2: Let $d_1 \in N^{(0)}$ and $d_2 \in N^{(1)}$. Thus $\overline{sx}_{(M+1)+L(M+2)} = d_1 \otimes g_1$, with $g_1 \in G^{(0)}$ and $\overline{sx}_{(L+1)(M+2)} = d_2 \otimes g_2$, with $g_2 \in G^{(1)}$. Since $G^{(0)} = Z(G)$ the proof is similar as in case 1 and also $[d_1, d_2] = 0$.

CASE 3: Let $d_1 \in N^{(1)}$ and $d_2 \in N^{(0)}$. We have the same conclusion as in case 2.

CASE 4: Let $d_1, d_2 \in N^{(1)}$. In this case $g_1, g_2 \in G^{(1)}$. Hence

$$e_2^*(\overline{sx}, \overline{sy}) = ((L+1)!)^M \cdot 2^L \cdot \binom{k+l+2}{2} \cdot [(e_{ij}(d_1 \circ d_2)) \otimes g] = 0,$$

where $\alpha \circ \beta = \alpha\beta + \beta\alpha$ is the Jacobi product, $g \in G$ and $i, j \in \{1, \ldots, k+l\}$. Then $d_1 d_2 + d_2 d_1 = 0$ for all $d_1, d_2 \in N^{(1)}$. In particular, if $d_1 = d_2$, we have $d_1^2 = 0$.

Thus the lemma is proved. ∎

LEMMA 4: *If $N^\sharp$ denotes the algebra obtained from N by adjoining a unit element, then*

$$T(G(M_{k,l}(F) \otimes N^\sharp)) = T(G(M_{k,l}(F))).$$

Proof: The conclusion $T(G(M_{k,l}(F) \otimes N^\sharp)) \subseteq T(G(M_{k,l}(F)))$ is trivial because $1 \in N^\sharp$. We want to show that $T(G(M_{k,l}(F)) \subseteq T(G(M_{k,l}(F) \otimes N^\sharp)))$. Now, let $f(x_1, \ldots, x_n)$ be a multilinear polynomial in $T(G(M_{k,l}(F))$ which is not an identity of $G(M_{k,l}(F) \otimes N^\sharp)$. Then there exist $\alpha_1, \ldots, \alpha_n \in G(M_{k,l}(F) \otimes N^\sharp)$ such that $f(\alpha_1, \ldots, \alpha_n) \neq 0$. We may clearly assume that there exist $\beta_1, \ldots, \beta_n \in M_{k,l}(F)^{(0)} \cup M_{k,l}(F)^{(1)}$ and $\gamma_1, \ldots, \gamma_n \in ((N^\sharp)^{(0)} \otimes G^{(0)} \dot{+} (N^\sharp)^{(1)} \otimes G^{(1)}) \cup ((N^\sharp)^{(1)} \otimes G^{(0)} \dot{+} (N^\sharp)^{(0)} \otimes G^{(1)})$ such that

$$f(\beta_1 \otimes \gamma_1, \ldots, \beta_n \otimes \gamma_n) \neq 0. \tag{1}$$

Let $\gamma_1, \ldots, \gamma_r \in (N^\sharp)^{(0)} \otimes G^{(0)} \dot{+} (N^\sharp)^{(1)} \otimes G^{(1)}$, $\beta_1, \ldots, \beta_r \in M_{k,l}(F)^{(0)}$ and $\gamma_{r+1}, \ldots, \gamma_n \in (N^\sharp)^{(1)} \otimes G^{(0)} \dot{+} (N^\sharp)^{(0)} \otimes G^{(1)}$, $\beta_{r+1}, \ldots, \beta_n \in M_{k,l}(F)^{(1)}$. Then we may assume that $\gamma_i = n_i^0 \otimes g_i^0 + n_i^1 \otimes g_i^1$, for $i = 1, \ldots, r$ and $\gamma_j = n_j^1 \otimes g_j^0 + n_j^0 \otimes g_j^1$, for $j = r+1, \ldots, n$. From (1) we have

$$\begin{aligned} 0 &\neq f(\beta_1 \otimes \gamma_1, \ldots, \beta_n \otimes \gamma_n) \\ &= f(\beta_1 \otimes (n_1^0 \otimes g_1^0 + n_1^1 \otimes g_1^1), \ldots, \beta_n \otimes (n_n^1 \otimes g_n^0 + n_n^0 \otimes g_n^1)) \\ &= \sum_k f(\beta_1 \otimes \delta_{1k}^0, \ldots, \beta_r \otimes \delta_{rk}^0, \beta_{r+1} \otimes \delta_{r+1k}^1, \ldots, \beta_n \otimes \delta_{nk}^1), \end{aligned}$$

where $\delta_{ik}^0 \in \{n_i^0 \otimes g_i^0, n_i^1 \otimes g_i^1\}$, $i = 1, \ldots, r$, and $\delta_i^1 \in \{n_i^1 \otimes g_i^0, n_i^0 \otimes g_i^1\}$, $i = r+1, \ldots, n$. Hence there exists k such that

$$\hat{f}_k = f(\beta_1 \otimes \delta_{1k}^0, \ldots, \beta_n \otimes \delta_{nk}^1) \neq 0.$$

More precisely, we have

$$\begin{aligned} 0 &\neq f(\beta_1 \otimes \delta_{1k}^0, \ldots, \beta_n \otimes \delta_{nk}^1) \\ &= f(\beta_1 \otimes (n_1^{i_1} \otimes g_1^{j_1}), \ldots, \beta_r \otimes (n_r^{i_r} \otimes g_r^{j_r}), \ldots, \beta_n \otimes (n_n^{i_n} \otimes g_n^{j_n})), \end{aligned}$$

where $i_k, j_k \in \{0, 1\}$, and $j_k = i_k$ for $k = 1, \ldots, r$, $j_k \neq i_k$ for $k = r+1, \ldots n$. Since from Lemmas 2 and 3 the n_i^0's commute with any elements and the n_i^1's anticommute among themselves, we can write

$$(2) \qquad 0 \neq f(\beta_1 \otimes \delta_{1k}^0, \ldots, \beta_n \otimes \delta_{nk}^1) = b \otimes (n_1 \cdots n_n) \otimes (g_1 \cdots g_n),$$

with $0 \neq b \in M_{k,l}(F)$ and $0 \neq n_1 \cdots n_n \otimes g_1 \cdots g_n \in N^\sharp \otimes G$.

Now, if we substitute in (2) the elements δ_{ik}^0 with distinct $g_i^0 \in G^{(0)}$ for $k = 1, \ldots, r$ and δ_{ik}^1 with $g_i^1 \in G^{(1)}$ for $k = r+1, \ldots, n$, then also

$$f(\beta_1 \otimes g_1^0, \ldots, \beta_r \otimes g_r^0, \beta_{r+1} \otimes g_{r+1}^1, \ldots, \beta_n \otimes g_n^1) = b \otimes g \neq 0$$

for the same $b \in M_{k,l}(F)$ and $0 \neq g = g_1^0 \cdots g_r^0 \cdot g_{r+1}^1 \cdots g_n^1 \in G$. Hence f is not an identity of $G(M_{k,l}(F))$ and the proof is complete. ∎

THEOREM 5: *Let $k, l \in \mathbb{N}$. Then* $\mathrm{var}(E^*_{k^2+l^2,2kl}) = \mathrm{var}(M_{k,l}(G) \oplus G(D'))$, *where D' is a finite-dimensional superalgebra such that* $\exp(D') < (k+l)^2$. *In particular,*

$$c_n(E^*_{k^2+l^2,2kl}) \simeq c_n(M_{k,l}(G)).$$

Proof: Berele and Regev in [3] proved that

$$\exp(E^*_{k^2+l^2,2kl}) = k^2 + l^2 + 2kl = (k+l)^2.$$

Moreover, by [9, Theorem 1], there exist some finite-dimensional reduced superalgebras $A_1, \ldots, A_s$ such that

$$\mathcal{V}_{k^2+l^2,2kl} = \mathrm{var}(G(A_1) \oplus \cdots \oplus G(A_s) \oplus G(D)), \tag{3}$$

where $\exp(G(A_1)) = \cdots = \exp(G(A_s)) = \exp(\mathcal{V}_{k^2+l^2,2kl}) = (k+l)^2$ and $\exp(G(D)) < \exp(\mathcal{V}_{k^2+l^2,2kl})$.

Now, we analyze the structure of a finite-dimensional reduced superalgebra A which satisfies $E^*_{k^2+l^2,2kl}$.

Let A be a finite-dimensional reduced superalgebra such that $\exp(G(A)) = \exp(\mathcal{V}_{k^2+l^2,2kl})$ and $E^*_{k^2+l^2,2kl} \subseteq T(G(A))$. We can write $A = B_1 \oplus \cdots \oplus B_q \dot{+} J$, where B_i are simple subalgebras and $J = J(A)$ is the Jacobson radical of A.

Recall that a simple finite-dimensional superalgebra B_i over F is isomorphic to one of the following algebras (see [11]):

1. $M_{d_i}(F)$, with trivial grading $(M_{d_i}(F), 0)$;
2. $M_{s_i}(D)$, where $D = F \oplus tF$ and $t^2 = 1$, with grading $(M_{s_i}(F), tM_{s_i}(F))$;
3. $M_{k_i,l_i}(F)$ with grading $\left(\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}, \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix}\right)$, where F_{11}, F_{12}, F_{21}, F_{22} are $k_i \times k_i$, $k_i \times l_i$, $l_i \times k_i$ and $l_i \times l_i$ matrices, respectively, $k_i > 0$ and $l_i > 0$.

Hence

$$G(A) = G(B_1) \oplus \cdots \oplus G(B_q) \dot{+} (J^{(0)} \otimes G^{(0)} \dot{+} J^{(1)} \otimes G^{(1)}),$$

where $G(B_i)$ is isomorphic to $M_{d_i}(F)$ or $M_{s_i}(G)$ or $M_{k_i,l_i}(G)$.

Let t_1 be the number of superalgebras B_i of the first type, let t_2 be the number of superalgebras B_i of the second type, and finally let t_3 be the number of B_i of the third type, $t_1 + t_2 + t_3 = q$. Then by [8] and [6] there exists a minimal (see definition in [8]) superalgebra C such that $G(C) \subseteq G(A)$ and

$$T(G(C)) = I_1 \cdots I_p,$$

where $I_i = T(G(D_i))$ and $\exp(G(C)) = \exp(G(A)) = (k+l)^2$. Hence $G(A)$ contains a subalgebra isomorphic to the following upper block triangular matrix algebra:

$$UT_p(r_1, \ldots, r_p) = \begin{pmatrix} G_{11} & & & * \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & \ldots & 0 & G_{pp} \end{pmatrix},$$

where $G_{ii} = G(D_i)$ is one of the following: $M_{d_i}(F)$, $M_{s_i}(G)$, $M_{k_i,l_i}(G)$; and $r_i = d_i$ or $r_i = s_i$ or $r_i = k_i + l_i$.

Since $G(C) \subseteq G(A)$, we have that $E^*_{k^2+l^2,2kl} \subseteq T(G(C))$.

Moreover, it is well known (see [14]) that the n-th cocharacter of the matrix algebra $M_d(F)$ lies in a strip of height d^2. Also, it is clear that the n-th cocharacter of the verbally prime algebra $M_s(G)$ lies in a hook of arm and leg s^2 and the n-th cocharacter of $M_{k,l}(G)$ lies in a hook of arm k^2+l^2 and leg $2kl$ (by virtue of [1, Theorem B], it is enough to check that $M_s(G)$ satisfies $e^*_{s^2,s^2}=0$ and $M_{k,l}(G)$ satisfies the identity $e^*_{k^2+l^2,2kl}=0$, which is evidently true). By applying the Littlewood–Richardson rule, Berele and Regev in [2, Theorem 1.1] give a rule to calculate the n-th cocharacter of a product of T-ideals. By this rule and by the results about the form of the n-th cocharacter of the verbally prime algebras mentioned before, similar to [3, 9] we can estimate the size of a hook and a square containing all diagrams which appear with non-zero multiplicity in the decomposition of the n-th cocharacter of $T(G(C))$. More precisely,

$$\chi_n(T(G(C))) = \sum_{\substack{\lambda \vdash n \\ \lambda \in H'}} m_\lambda \chi_\lambda$$

where $H' = H(k^2+l^2, 2kl) \cup D(k^2+l^2+m, 2kl+m)$ is the hook of arm k^2+l^2 and leg $2kl$ plus a rectangle of size $m^2 > (p-1)^2$. In this decomposition, because we have $p-1$ multiplication, there is one diagram $D_{\hat{\mu}}$ with non-zero multiplicity and containing the rectangle of size $(2kl+1)^{k^2+l^2+p-1}$. Then, by [1, Theorem B], $E^*_{k^2+l^2+p-2,2kl} \not\subseteq T(G(C))$. Taking into account $E^*_{k^2+l^2,2kl} \subseteq T(G(C))$ we conclude $p \leq 1$. It means $p=1$ and $C \in \{M_{d_i}(F), M_{s_i}(D), M_{k_i,l_i}(F)\}$. We have $G(C)$ is a subalgebra of the algebra $G(A)$, where A is reduced and $\exp(G(A)) = \exp(G(C))$. Then from [7] $\exp(G(A)) = \sum_{i=1}^q \dim B_i = \exp(G(C))$. Also, granting B_i and C are simple superalgebras we obtain $q=1$ and can assume $B_1 \cong C$.

Thus $A = B + J$ with $B \cong M_{d_1}(F)$ or $B \cong M_{s_1}(D)$ or $B \cong M_{k_1,l_1}(F)$. Since $\exp(G(A)) = (k+l)^2$ and $G(A)$ corresponds to the hook $H(k^2+l^2, 2kl)$ for $l \neq 0$ (see [7]), we have $k_1 = k$, and $l_1 = l$. Then

$$A \cong M_{k,l}(F) \dot{+} (J^{(0)} \dot{+} J^{(1)}),$$

and

$$G(A) \cong M_{k,l}(G) \dot{+} (J^{(0)} \otimes G^{(0)} \dot{+} J^{(1)} \otimes G^{(1)})$$

with $G(A) \in \mathcal{V}_{k^2+l^2,2kl}$. From Lemmas 1, 2 and 3 we have

$$A \cong (M_{k,l}(F) + J_{11}) \oplus J_{00} \cong (M_{k,l}(F) \otimes N^\sharp) \oplus J_{00}.$$

From Lemma 4, $T(G(M_{k,l}(F) \otimes N^{\sharp})) = T(G(M_{k,l}(F)))$, then $\mathrm{var}(G(A)) = \mathrm{var}(M_{k,l}(G) \oplus G(J_{00}))$; here J_{00} is nilpotent. Hence, recalling the decomposition given in (3), we get

$$\mathcal{V}_{k^2+l^2,2kl} = \mathrm{var}(M_{k,l}(G) \oplus G(D')),$$

where D' is a finite-dimensional superalgebra with $\exp(G(D')) < (k+l)^2$. Then, from [9, Corollary 2] we have

$$c_n(E^*_{k^2+l^2,2kl}) \simeq c_n(M_{k,l}(G)). \quad \blacksquare$$

3. Asymptotics for $E^*_{s^2,s^2}$ and $M_s(G)$

In this section we shall prove that the codimensions of Γ_{s^2,s^2} are asymptotically equal to the codimensions of the verbally prime algebra $M_s(G)$, $s \in \mathbb{N}$. Throughout this section we assume that $A = M_s(D) \dot{+} J$, where $M_s(D)$ is the simple reduced superalgebra of $s \times s$ matrices over $D = F \oplus tF$ ($t^2 = 1$) with grading $(M_s(F), tM_s(F))$ and $J = J(A)$ is the Jacobson radical of the finite-dimensional superalgebra A. We start with the following lemma, which establishes a similar result to Lemma 1.

LEMMA 6: *The Jacobson radical J can be decomposed into the direct sum of four $M_s(D)$-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $p, q \in \{0, 1\}$, J_{pq} is a left faithful module or a 0-left module according as $p = 1$ or $p = 0$, respectively. Similarly, J_{pq} is a right faithful module or a 0-right module according as $q = 1$ or $q = 0$, respectively. Moreover, for $p, q, i, l \in \{0, 1\}$, $J_{pq}J_{ql} \subseteq J_{pl}$, $J_{pq}J_{il} = 0$ for $q \neq i$ and there exists a finite-dimensional nilpotent superalgebra N such that $J_{11} \cong M_s(D) \otimes_F N$ (isomorphism of $M_s(D)$-bimodules and of superalgebras).

Proof: Similar to the proof of Lemma 1 the first part follows from the [9, Lemma 2]. We only should note that we can choose a homogeneous system Q of elements $j \in J_{11}$ generating the J_{11} as a $M_s(D)$-bimodule. Let $Q = Q^{(0)} \cup Q^{(1)}$, where $Q^{(0)}$ contains all even elements of Q and $Q^{(1)}$ contains the odd ones. Then we consider as in Lemma 1 the elements $d_{km}(j) = \sum_{i=1}^{s} e_{ik} j e_{mi}$, $k, m = 1, \ldots, s$, $j \in Q$, which has the same grading type as j and commutes with elements of $M_s(D)$. Take $N = N^{(0)} \oplus N^{(1)}$, where

$$N^{(0)} = \mathrm{Span}\{d_{km}(j) \colon j \in Q^{(0)}, k, m = 1, \ldots, s\}$$

and

$$N^{(1)} = \mathrm{Span}\{d_{km}(j): j \in Q^{(1)}, k, m = 1, \ldots, s\}.$$

We have that $M_s(D) \otimes N$ is a superalgebra with natural $\mathbb{Z}_2$-grading: $(M_s(F) \otimes N^{(0)} \dot{+} M_s(F) \cdot t \otimes N^{(1)}, M_s(F) \otimes N^{(1)} \dot{+} M_s(F) \cdot t \otimes N^{(0)})$, and the D-linear map $\phi: J_{11} \to M_s(D) \otimes N$, $\phi: e_{rk} j e_{mq} \mapsto e_{rq} \otimes d_{km}(j)$ is extended to an isomorphism of the superalgebras. The lemma is proved. ∎

Now, we will use the polynomials $e_1^*(\overline{x}; \overline{y})$ and $e_2^*(\overline{x}; \overline{y})$ from Lemmas 2 and 3 to prove the following two lemmas.

LEMMA 7: *Let $M = L = s^2$ with $s \in \mathbb{N}$. If $E^*_{M,L} \subseteq Id(G(A))$, then $J_{10} = J_{01} = (0)$.*

Proof: For $M = L = s^2$ we construct, as in Lemma 2, the polynomial $e_1^*(\overline{x}; \overline{y})$ as a consequence of $E^*_{M,L}$. Now, we make the following substitution: let $e_1, \ldots, e_s$ be an ordered basis of $M_s(F)$ consisting of all matrix units, for $1 \le i \le M$ and $1 \le j \le L+1$. We set

$$\overline{sx}_{i+(j-1)(M+1)} := e_i \otimes g^0_{ji}, \quad j \neq i$$

and

$$\overline{sx}_{i+(i-1)(M+1)} := e_i \otimes g^1_{ii}$$

where $g^0_{ji} \in G^{(0)}$ and $g^1_{ii} \in G^{(1)}$. Also, we get

$$\overline{sx}_{j(M+1)} := e_j \otimes g^1_{j(M+1)}, \quad 1 \le j \le L$$

and

$$\overline{sx}_{(L+1)(M+1)} := r_{10} \otimes g$$

where $g^1_{j(M+1)} \in G^{(1)}$, $r_{10} \in J_{10}$ and $g \in G^{(0)} \cup G^{(1)}$.

As in Lemma 2 we put instead of the y's the elements of the type $e \otimes g_0$, where e is a matrix unit and $g_0 \in G^{(0)}$. Choosing the matrix units e we fix all places for $\overline{sx}$'s.

As a result, after this substitution of elements of $G(A)$, we obtain

$$e_1^*(\overline{sx}, \overline{sy}) = (L!)^M 2^L e_{ij} r_{10} \otimes g = 0,$$

where $g \in G$. Thus $r_{10} = 0$ for all $r_{10} \in J_{10}$. Analogously, $J_{01} = 0$ and the lemma is proved. ∎

LEMMA 8: *Let $M = L = s^2$ with $s \in \mathbb{N}$. Let $J_{11} \simeq M_s(D) \otimes N$, where $N = N^{(0)} \oplus N^{(1)}$, as in Lemma 6. If $E^*_{M,L} \subseteq Id(G(A))$, then $N^{(0)} \subseteq Z(N)$, the center of N, and $N^{(1)}$ is anticommutative (or, which is the same, is nil of degree 2).*

Proof: As in Lemma 3 for $M = L = s^2$ we determine the polynomial $e_2^*(\overline{x}, \overline{y})$, which is a consequence of $E^*_{M,L}$. Then we make the following substitution, similar to the substitution in Lemma 7: let $e_1, \ldots, e_s$ be an ordered basis consisting of matrix units of $M_s(F)$, for $1 \le i \le M$ and $1 \le j \le L+1$. We put

$$\overline{sx}_{i+(j-1)(M+1)} := e_i \otimes g^0_{ji}, \quad j \neq i$$

and

$$\overline{sx}_{i+(i-1)(M+1)} := e_i \otimes g^1_{ii}$$

where $g^0_{ji} \in G^{(0)}$ and $g^1_{ii} \in G^{(1)}$ and they are all distinct. Moreover, we put

$$\overline{sx}_{(M+1)+(j-1)(M+2)} := e_j \otimes g^0_{j(M+1)}, \quad 1 \le j \le L,$$
$$\overline{sx}_{j(M+2)} := e_j \otimes g^1_{j(M+2)}, \quad 1 \le j \le L$$

and

$$\overline{sx}_{(M+1)+L(M+2)} := d_1 \otimes g_1,$$
$$\overline{sx}_{(L+1)(M+2)} := d_2 \otimes g_2,$$

where $g^0_{j(M+1)} \in G^{(0)}$ and $g^1_{j(M+2)} \in G^{(1)}$ are all distinct, $g_1, g_2 \in G^{(0)} \cup G^{(1)}$ and $d_1, d_2 \in N^{(0)} \cup N^{(1)}$. Also, as in the previous lemmas, we substitute the y's with elements of the type $e \otimes g_0$, where the e's are distinct matrix units and $g_0 \in G^{(0)}$ taken to fix all places for $\overline{sx}$'s. Hence we obtained the following results: if $d_1, d_2 \in N^{(0)}$ or $d_1 \in N^{(0)}$ and $d_2 \in N^{(1)}$ or $d_1 \in N^{(1)}$ and $d_2 \in N^{(0)}$, then similarly to [9, Lemma 4]

$$e_2^*(\overline{sx}, \overline{sy}) = K_1 (L!)^M 2^L e_{ii}[d_1, d_2] \otimes g = 0,$$

for some $g \in G$ and some natural K_1. Thus $[d_1, d_2] = 0$. If $d_1, d_2 \in N^{(1)}$ then

$$e_2^*(\overline{sx}, \overline{sy}) = K_2 (L!)^M 2^L (e_{ii}(d_1 \circ d_2)) \otimes g = 0, \quad K_2 \in \mathbb{N}.$$

Hence $d_1 d_2 + d_2 d_1 = 0$ for all $d_1, d_2 \in N^{(1)}$ and the lemma is proved. ∎

The proof of the following lemma is the same as the proof of Lemma 4.

LEMMA 9: *If $N^\sharp$ denotes the algebra obtained from N by adjoining a unit element, then*

$$T(G(M_s(D) \otimes N^\sharp)) = T(G(M_s(D))).$$

THEOREM 10: *Let $s \in \mathbb{N}$, $s > 0$. Then* $\mathrm{var}(E^*_{s^2,s^2}) = \mathrm{var}(M_s(G) \oplus G(D'))$, *where D' is a finite-dimensional superalgebra such that* $\exp(D') < 2s^2$. *In particular,*

$$c_n(E^*_{s^2,s^2}) \simeq c_n(M_s(G)).$$

Proof: The first part of the theorem is the same as that of Theorem 5. Hence we have a finite-dimensional reduced superalgebra A with $\exp(G(A)) = \exp(\mathcal{V}_{s^2,s^2})$ and $E^*_{s^2,s^2} \subseteq T(G(A))$, and $A = B \dot{+} J$ with $B \cong M_{d_1}(F)$ or $B \cong M_{s_1}(D)$ or $B \cong M_{k_1,l_1}(F)$. Since, by a result of Berele and Regev [3], $\exp(G(A)) = \exp(\mathcal{V}_{s^2,s^2}) = 2s^2$, we have

$$A \cong M_s(D) \dot{+} (J^{(0)} \dot{+} J^{(1)}),$$

and

$$G(A) \cong M_s(G) \dot{+} (J^{(0)} \otimes G^{(0)} \dot{+} J^{(1)} \otimes G^{(1)})$$

with $G(A) \in \mathcal{V}_{s^2,s^2}$. By Lemmas 6, 7 and 8 we have

$$A \cong (M_s(D) \dot{+} J_{11}) \oplus J_{00} \cong (M_s(D) \otimes N^\sharp) \oplus J_{00}.$$

From Lemma 9, $T(G(M_2(D) \otimes N^\sharp)) = T(G(M_s(D)))$. Then $\mathrm{var}(G(A)) = \mathrm{var}(M_s(G) \oplus G(J_{00}))$, where J_{00} is nilpotent. Hence, we get

$$\mathcal{V}_{s^2,s^2} = \mathrm{var}(M_s(G) \oplus G(D')),$$

where D' is a finite-dimensional superalgebra with $\exp(G(D')) < 2s^2$. So, by [9, Corollary 2], we have

$$c_n(E^*_{s^2,s^2}) \simeq c_n(M_s(G))$$

and the proof is complete. ∎

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